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Gaugino condensation and generation of supersymmetric 3-form flux

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Abstract: We extend the linearised solution of Polchinski and Strassler describing the supergravity dual of the $\mathcal{N} = 1^*$ gauge theory. By analysing the equations of motion of type IIB supergravity at cubic order in the mass perturbation parameter, we demonstrate the emergence of a 3-form flux of type $(3,0)$ with respect to the natural complex structure. The generation of this flux can be associated to the dynamical formation of a gaugino condensate in the confining phase of the $\mathcal{N} = 1^*$ gauge theory. We also check that the supersymmetry conditions are satisfied, and we discuss how this $(3,0)$-form flux is tied to the existence of a supersymmetric background with SU(2)-structure.

Keywords: AdS-CFT and dS-CFT Correspondence, Supersymmetry and Duality.

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Contents

1. Introduction

The AdS/CFT correspondence [1]–[3] states that certain (supersymmetric) quantum field theories have a dual description in terms of (super)gravity theories on backgrounds which contain AdS factors. An example thereof is provided by type IIB supergravity on an $AdS_5 \times S^5$ background with N units of Ramond 5–form flux, which is conjectured to be dual to $\mathcal{N} = 4$ supersymmetric SU(N) gauge theory at large N and at large 't Hooft coupling $g_{YM}^2 N$. This gauge theory, which is conformal, lives on a stack of N D3-branes. In subsequent studies, this conjecture has been generalised to a correspondence between non–conformal gauge theories with less supersymmetry and supergravity theories on deformed backgrounds. An example of the latter is the type IIB supergravity solution of Polchinski and Strassler [4], which describes the supergravity dual of the $\mathcal{N} = 1^*$ SU(N) gauge theory which is obtained from the $\mathcal{N} = 4$ SU(N) gauge theory by mass deformation, namely by giving mass m to all three adjoint chiral superfields.

The $\mathcal{N} = 1^*$ theory is interesting because it possesses confining phases [5]–[9]. Moreover, for this theory the AdS/CFT correspondence allows for a quantitative description of various non–perturbative field theory phenomena such as flux tubes, baryon vertices, domain walls, instantons and condensates [4]. In this paper, we will be concerned with the description of gaugino condensation. As soon as the mass perturbation m is turned on, a dynamical gaugino condensate $\langle \bar{\lambda}\bar{\lambda} \rangle \sim \Lambda^3$ gets formed in the confining phase. Since, at large 't Hooft coupling $\Lambda = m$, the formation of a gaugino condensate is an order m^3-effect in the dual supergravity description. According to the AdS/CFT correspondence, the formation of a gaugino condensate (as well as its mass) corresponds, in the dual supergravity solution, to the generation of a 2–form potential with polarisation tensor ε_{ijk} at order m^3 [4]. It was shown in [4] that the linearised equations of motion do indeed allow for such a solution, albeit with an undetermined coefficient in front of it, whose precise value is in principle determined by the infrared physics. On the other hand, when going beyond the linearised approximation and thereby extending the analysis of [4], additional solutions are expected to arise at order m^3. These solutions, being solutions to inhomogeneous equations, are uniquely determined. It is in these inhomogeneous solutions that we will be interested in this paper. They are the ones connected to the generation of 3–form flux in the Polchinski-Strassler solution associated with the dynamical formation of a gaugino condensate[1] $\langle\bar{\lambda}\bar{\lambda}\rangle$ in the confining phase of the $\mathcal{N}=1^*$ gauge theory.

Inspired by analogy with compactifications of heterotic string theory and of type IIB string theory on Calabi–Yau manifolds in the presence of non-trivial fluxes [10], we will be looking for the (3,0)-component of the 3-form flux G_3. In heterotic string theory, when considering a Minkowski ground state, the formation of a gaugino condensate [11]–[18] requires the presence of a compensating 3-form H-flux of Hodge types $(3,0)$ and $(0,3)$ (see also [19]).[2] On the other hand, in type IIB compactifications with D3-branes [27, 28], it turns out that the effective soft SUSY breaking terms originating from (3,0)-fluxes are of the type arising in dilaton dominated SUSY breaking scenarios. The latter also naturally arise in the context of spontaneous supersymmetry breaking by gaugino condensation. Actually, in [27, 28], a direct link between the (3,0)-components of the fluxes and the mass of the gaugini is established, and this is suggestive of a similar relation for the gaugino condensate. Of course, in heterotic string compactifications as well as in type IIB theory on compact (warped) Calabi-Yau spaces, gravity cannot be decoupled as in the AdS/CFT context, so that the resulting effective theories are *locally* supersymmetric field theories in which the formation of a gaugino condensate, combined with the requirement of vanishing cosmological constant, results in the spontaneous breaking of $\mathcal{N}=1$ supersymmetry.

In this paper, on the other hand, we will be dealing with a *globally* supersymmetric field theory. We will show that the formation of a gaugino condensate in such a field theory can be related, through the AdS/CFT correspondence, to the emergence of a (3,0) G_3-flux in type IIB supergravity. To be specific, we will show that when going beyond the linearised analysis of the equations of motion given in [4], a $(3,0)$-piece in the field strength G_3 *does* get generated to order m^3. We will show that we may associate a 2-form potential with polarisation tensor ε_{ijk} to it, thereby associating the emergence of this $(3,0)$-flux to the formation of a gaugino condensate. This $(3,0)$-form flux arises when taking into account that the axion/dilaton field τ ceases to be constant at order m^2 [29], and solving

[1]Because of the non–trivial superpotential and consequent operator mixing, the operator growing a vev is actually a linear combination of $\lambda\lambda$, Φ^3 and $m\Phi^2$ defining a primary operator orthogonal to the combination appearing in the long Konishi multiplet [41].

[2]Additional supersymmetry preserving H–flux of Hodge types $(2,1)$ and $(1,2)$ may be needed to compensate for non-Kähler deformations compatible with an SU(3)-structure of the compactification manifold [20]–[26].

the associated inhomogeneous equations of motion for G_3. Note that at order m^3, the solution of the linearised equations of motion alluded to above [4] does *not* give rise to a 3-form flux of Hodge type $(3,0)$. Therefore, at order m^3, it is only when going beyond the linearised approximation that we see the emergence of an imaginary anti-selfdual 3-form flux component $G^{(3,0)}$ of Hodge type $(3,0)$.

Since the $\mathcal{N} = 1^*$ gauge theory is a *globally* supersymmetric field theory, the ground state described by the gaugino condensate is $\mathcal{N} = 1$ supersymmetric. For this reason, and since the infrared vacuum is not conformal, the associated 3-form flux on the dual supergravity side is expected to be part of a supergravity background preserving $\mathcal{N} = 1$ supersymmetry. This means that, in contrast to the (non-compact) Calabi-Yau solutions, the emergence of $G^{(3,0)}$ should be compatible with an $\mathcal{N} = 1$ residual supersymmetry of the background. This may raise the issue of the consistency of such a background, since supersymmetry is linked to the Hodge type of the 3-form flux and, for solutions preserving 4-dimensional Poincaré invariance, the flux is usually of type (2,1) and/or (1,2). This fact is tied to a specific spinor ansatz though, which is not general enough to capture the above solution. This ansatz reads

$$\epsilon(x,y) = a(y)\,\varepsilon(x) \otimes \eta_-(y) + b(y)\,\varepsilon^*(x) \otimes \eta_+(y)\,, \tag{1.1}$$

where a and b denote complex functions, ε is the four-dimensional supersymmetry parameter and $\eta_+ = (\eta_-)^*$ is a globally defined spinor normalised to one. The existence of one globally defined spinor η implies that the tangent bundle over the transverse 6-dimensional space has an SU(3) group structure. The ansatz (1.1) includes the type A ansatz [20], where $b = a^*$, the type B [30]–[32], where $b = 0$, and the more general type discussed in [33], which we call type C. In any of these cases, the allowed 3-form flux is constrained to contain only (2,1) and/or (1,2) fluxes [34, 35]. One may then ask how a 3-form flux of Hodge type $(3,0)$, like the one which arises at order m^3 in the Polchinski-Strassler background, can be compatible with $\mathcal{N} = 1$ supersymmetry. The solution to this question resides in using an even more general supersymmetry ansatz, called type D [34]. This ansatz reads

$$\epsilon(x,y) = a(y)\,\varepsilon(x) \otimes \eta_-(y) + \varepsilon^*(x) \otimes \left(b(y)\,\eta_+(y) + c(y)\,\chi_+(y)\right). \tag{1.2}$$

It is based on the existence of two globally defined spinors, η and χ, which are linearly independent. It implies that the group structure of the tangent bundle of the transverse 6-dimensional space is further reduced to SU(2). Although this requirement is stronger than the one used to formulate (1.1), it is necessary for obtaining more general solutions, as it was already observed in type IIA [36] and when considering AdS_5 solutions in M-theory or type IIB [37]. In particular, using the ansatz (1.2), the Hodge type of the 3-form flux is no more constrained by supersymmetry, and one can now have (3,0) as well as (0,3) fluxes [34]. Moreover, and in contrast with the case of SU(3)-structures, there is no preferred choice for the (almost) complex structure J, but one actually has a U(1)-worth of possibilities. This would make the computation of the Hodge type of the 3-form pointless. However, for the solution at hand, there is a way to fix J, namely by the fact that in the ultraviolet regime we should recover the $AdS_5 \times S^5$ solution, which describes the near-horizon solution of a

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stack of D3-branes. This means that in the ultraviolet the spinor ansatz should reduce to the type B ansatz discussed above and therefore there is a unique choice of J associated to it.

There is an additional motivation for the fact that the ansatz D (1.2) is the appropriate one to describe the Polchinski-Strassler solution and that the transverse 6-dimensional space displays an SU(2)-structure. The addition of a mass perturbation to the chiral superfields Φ_i implies that the F-term equations for a supersymmetric vacuum read $[\Phi_i, \Phi_j] = -m\, \varepsilon_{ijk}\Phi_k$. Since these fields have to be interpreted as the transverse coordinates to the stack of D3-branes, this means that we may have vacua where the D3-branes are spread over the transverse space building up a granular S^2, i.e. they are polarised into 5-branes [38, 4]. This obviously affects the spinor ansatz, and in addition to the usual projector on ϵ coming from the presence of the D3-branes one should have an additional one consistent with the 5-branes. All in all, this means that one expects a supersymmetry projector of the form [39]

$$\mathcal{P} = \frac{1}{2}\left(1 - i\gamma^0\gamma^1\gamma^2\gamma^3\left(\cos\varphi + \sin\varphi\,\gamma^4\gamma^5 *\right)\right)\,, \qquad \mathcal{P}\epsilon = \epsilon\,, \tag{1.3}$$

where $*$ denotes complex conjugation. After some trivial algebra, it can be shown that this projector naturally selects a spinor of the form given in (1.2) with $b = 0$ (see also [34]). It is therefore natural to expect that the full solution will possess an SU(2)-structure.

Our findings may thus be summarised in the following way,

$$\begin{tikzcd} m^3 \arrow[r] & G^{(3,0)} \arrow[dl] \arrow[dr] & \\ \boxed{\text{SU(2)-structure}} & & \boxed{\text{polarisation } \varepsilon_{ijk}} \end{tikzcd}$$

The paper is organised as follows. In section 2 we review deformations of the AdS/CFT correspondence and their relation to the harmonic analysis presented in [40]. In section 3 we establish the presence of a $(3,0)$-piece in G_3 to order m^3, and we determine the associated 2-form potential. In section 4 we first compute the non-trivial behaviour of the axion/dilaton τ to second order in m. Then we briefly review the impact of source terms on the Polchinski-Strassler solution. Next we turn to the explicit computation of various bulk contributions to G_3 to order m^3. In section 5 we show that the emergence of a $(3,0)$-piece in G_3 is compatible with supersymmetry based on the type D spinor ansatz. We present our conclusions in section 6. The appendix contains a summary of useful expressions for the Polchinski-Strassler solution.

2. Deformations of the AdS/CFT correspondence

In this section we review the dictionary between field theory operators and supergravity fields so that we can correctly identify the couplings and vacuum expectation values (vevs) which are turned on in our solution.

Consider an operator $\mathcal{O}$ of mass dimension Δ. It may be added to a four-dimensional CFT hamiltonian,

$$H = H_{\mathrm{CFT}} + a\,\mathcal{O}\,. \tag{2.1}$$

The coupling a has mass dimension $4-\Delta$. The operator $\mathcal{O}$ may also develop a vev

$$\langle 0|\mathcal{O}|0\rangle = b\,. \tag{2.2}$$

The vev b has mass dimension Δ. In the dual supergravity description, a and b are read off from the radial behaviour of the corresponding supergravity field at large radius r,

$$a\left(\frac{r}{R}\right)^{\Delta-4} + b\,R^{2\Delta-4}\left(\frac{r}{R}\right)^{-\Delta}. \tag{2.3}$$

Here $r^{\Delta-4}$ denotes the non-normalisable solution, whereas $r^{-\Delta}$ is the normalisable one. $R=(4\pi g N\alpha'^2)^{1/4}$ denotes the radius of AdS_5. Here, large r corresponds to the ultraviolet regime in the dual field theory.

The supergravity solution we will be considering is the type IIB solution describing the supergravity dual of the $\mathcal{N}=1^*$ theory of Polchinski and Strassler [4]. The $\mathcal{N}=4$ supersymmetry of the unperturbed SU(N) gauge theory gets broken down to $\mathcal{N}=1^*$ by turning on mass terms for all the three chiral superfields Φ_i. In the dual type IIB supergravity solution this corresponds to turning on a 2-form potential with asymptotic behaviour (2.3), whose field strength is related to the 3-form flux $G_3 = F_3 - \tau H_3$. Here $F_3 = dC_2$ and $H_3 = dB_2$ denote the RR and the NSNS 3-form field strengths, respectively, and $\tau = C + i\mathrm{e}^{-\Phi}$ denotes the type IIB axion/dilaton. As argued in the introduction, also the gaugino condensate couples to the 2-form potential and this gives rise to an additional contribution to the 3-form flux. Let us then describe the correspondence for the 2-form potential in more detail.

At large r, the supergravity solution goes back to $AdS_5\times S^5$. For this background the AdS/CFT correspondence has been explored in great detail and all the possible field theory operators $\mathcal{O}$ have been matched [3] with the Kaluza-Klein states appearing in the analysis of [40]. For our purpose, this implies that at least at the linearised level we can read from the behaviour at large r which couplings (2.1) and vevs (2.2) get generated in the $\mathcal{N}=1^*$ theory. The relation between the string quantities used here such as G_3 and the supergravity variables used in the KK reduction [40] of type IIB supergravity theory can be read off from[3]

$$\kappa\, G_{SU} = i g^{1/2}\,\tau_2^{-1/2}\,\mathrm{e}^{i\theta}\,G_3\,, \qquad \mathrm{e}^{i\theta} = \left(\frac{1+i\bar\tau}{1-i\tau}\right)^{1/2}, \tag{2.4}$$

where $G_{SU} = f(dA_2 - Bd\bar{A}_2)$ and $\kappa\,A_2 = g^{1/2}(B_2 + iC_2)$. Here $B=(1+i\tau)/(1-i\tau)$ and $f^{-2} = 1-|B|^2$. For a constant axion/dilaton, i.e. $\tau=\tau_0 = C_0 + ig^{-1}$, so that $\tau_2 = g^{-1}$, we obtain the following relation between potentials

$$f\,\hat{A}_2 - f B\overline{\hat{A}_2} = i\,\mathrm{e}^{i\theta}\,(C_2 - \tau_0 B_2)\,, \qquad \hat{A}_2 = B_2 + iC_2\,. \tag{2.5}$$

[3]Note that (2.4) differs by a factor of $\sqrt{g}$ from the expression given in [32].

Setting

$$C_2 - \tau_0 B_2 = r^p \, S_2 \,, \tag{2.6}$$

where S_2 denotes a 2-form, yields

$$f \, \hat{A}_2 - f B \overline{\hat{A}}_2 = i \, \mathrm{e}^{i\theta} \, r^p \, S_2 \,. \tag{2.7}$$

Inverting this relation we obtain

$$\hat{A}_2 = i \, r^p \, \left(\mathrm{e}^{i\theta} \, f \, S_2 - \mathrm{e}^{-i\theta} \, f B \, \bar{S}_2 \right) , \tag{2.8}$$

which expresses the supergravity potential $\hat{A}_2$ in terms of S_2 and $\bar{S}_2$. Following [4], the boundary operators we are interested in couple to the 2-form potential $C_2 - \tau_0 B_2$. Nevertheless, we can use the harmonic analysis given in [40] to identify them, since this analysis was performed for $B = 0$, which implies that the expansion of $\hat{A}_2$ and of $C_2 - \tau_0 B_2$ is the same. This is validated by the fact that the 2-form $r^p S_2$ appearing in (2.7) is the lowest harmonic 2-form on S^5 and is given by

$$r^p S_2 = \frac{1}{2} \, r^{p+3} \, T_{mnp} \, \left(\frac{x^m}{r} \right) d \left(\frac{x^n}{r} \right) \wedge d \left(\frac{x^p}{r} \right) , \tag{2.9}$$

where T_{mnp} denotes a constant 3-form tensor in 6 dimensions. Comparison with (2.3) shows that this 2-form corresponds to an operator deformation when $p = \Delta - 7$, and to a vev of an operator when $p = -\Delta - 3$.

In the case of the Polchinski-Strassler solution, the 3-form $T_3 = \frac{1}{6} \, T_{mnp} dx^m \wedge dx^n \wedge dx^p$ is taken to satisfy $\star_6 T_3 = -i T_3$, which corresponds to the $\overline{\mathbf{10}}$ representation of the SO(6) tangent space group [4]. The associated 2-form potential (2.6) satisfies the linearised bulk equation of motion for $G_3 = d(C_2 - \tau_0 \, B_2)$,

$$d \left(Z^{-1} \, (\star_6 - i) G_3 \right) = 0 \,, \qquad Z = \frac{R^4}{r^4} \,, \tag{2.10}$$

provided that $p = -4, -6$ [4]. The 2-form potential is thus given by (2.9) with $p = -4, -6$, which describes the non-normalisable ($p = -4$) solution associated to turning on an operator of dimension $\Delta = 3$, as well as the normalisable ($p = -6$) solution associated with giving a vev to an operator of dimension $\Delta = 3$. This 2-form potential is a harmonic 2-form on S^5 with eigenvalue $M^2 = \Delta(\Delta - 4) = -3$. There are thus two homogeneous solutions to (2.10) given by $G_3 = 3 \, r^{-4} (T_3 - \frac{4}{3} \, V_3)$ and $G_3 = 3 \, r^{-6} (T_3 - 2 V_3)$, where $V_3 = d \log r \wedge S_2$ [4]. In the following, we will write the latter as $G_3 = 3 \, r^{-6} \left(\hat{T}_3 - 2 \hat{V}_3 \right)$. The constant tensor $\hat{T}_3$ can have different entries than T_3.

Let us now see which are the corresponding operators. Since T_3 is chosen in the $\overline{\mathbf{10}}$ of SU(4), the corresponding operators should be in the $\mathbf{10}$ and can be identified with the fermionic bilinears of the $\mathcal{N} = 4$ theory, namely $\mathcal{O}^{10} = \mathcal{O}^6_{ij} + \mathcal{O}^3_i + \mathcal{O}^1 = \psi_i \psi_j + \lambda \psi_i + \lambda \lambda$, where the splitting is done according to representations of the SU(3) subgroup of SU(4). The $\mathcal{N} = 1^*$ gauge theory is obtained by deforming the $\mathcal{N} = 4$ gauge theory by a mass deformation of the form $W = \frac{1}{2} m_{ij} \, \Phi_{(i} \Phi_{j)}$. The superfield bilinear $\Phi_{(i} \Phi_{j)}$ (as well as $\mathcal{O}_{ij}$) transforms as a $\mathbf{6}$ under the SU(3) subgroup of the SO(6) R-symmetry, whereas

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the mass matrix m_{ij} transforms as a $\bar{\mathbf{6}}$, and it is this coupling which will be described by T_3. The $\bar{\mathbf{6}}$ can be represented by the primitive $(1,2)$-part T_3 of an imaginary anti-selfdual 3-form, $\star_6 T_3 = -iT_3$. Therefore, turning on equal mass terms for all the three chiral superfields Φ_i corresponds, on the supergravity side, to turning on the $p = -4$ solution $G_3 = 3\,r^{-4}(T_3 - \frac{4}{3}\,V_3)$ with T_3 being proportional to the mass parameter m [4]. Consequently, only some of the components of $T_{i\bar{j}\bar{k}}$ are actually turned on. The component T_{ijk} is not turned on, since this would amount to turning on a mass for the gaugini, thereby breaking supersymmetry.

On the other hand, the $p = -6$ solution $G_3 = 3\,r^{-6}(\hat{T}_3 - 2\,\hat{V}_3)$ may get turned on at order m^3, since $\hat{T}_3 \sim m^3$ on dimensional grounds. As discussed above, it corresponds to turning on a vev of an operator of dimension $\Delta = 3$. Note that the constant tensor $\hat{T}_3$ can have different entries than the tensor T_3 appearing at linear order in m. $\hat{T}_3$ may, for instance, have a non-vanishing component $\hat{T}_{ijk} \propto \varepsilon_{ijk}$. Since $\hat{T}_3$ is in the $\overline{\mathbf{10}}$, the SU(3)-singlet $\hat{T}_{ijk}$ corresponds to a non-vanishing vev of the gaugino condensate $\overline{\mathcal{O}} = \bar{\lambda}\bar{\lambda}$ in the confining phase of the $\mathcal{N} = 1^*$ theory. This is so, because the bilinear $\lambda\lambda$ belongs to the $\mathbf{10}$, has mass dimension three and is a singlet under the SU(3)-subgroup of the SO(6) R-symmetry of the unperturbed $\mathcal{N} = 4$ theory. Its vev is given by $\Lambda^3 = m^3\,\exp(-8\pi^2/g^2_{\mathrm{YM}}N)$. In the large 't Hooft-coupling limit we then have $\Lambda^3 = m^3$, which is the behaviour of $\hat{T}_3$. Note that the operator growing a vev is actually a linear combination of $\bar{\lambda}\bar{\lambda}$, $\bar{\Phi}^3$ and $m\bar{\Phi}^2$, namely a primary operator orthogonal to the combination appearing in the Konishi multiplet [41]. By complex conjugation, also $\lambda\lambda$ grows a vev, which corresponds to the entries of $\overline{\hat{T}}_3$. The harmonic $\bar{S}_2$ in (2.8) is therefore associated with a vev of a linear combination of $\lambda\lambda$, Φ^3 and $m\Phi^2$. In the Higgs phase, on the other hand, there is no gaugino condensate and hence no gaugino condensate vev. However, a linear combination of Φ^3 and $m\Phi^2$ may have a non-vanishing vev (these two vevs are in turn get related through $\langle \mathrm{tr}\,\Phi\,\partial_\Phi W(\Phi)\rangle = 0$). Therefore, also in the Higgs phase a homogeneous solution (2.9) with $p = -6$ may arise with a non-vanishing coefficient in front of it.

Finally, observe that the solution $G_3 = 3\,r^{-6}\left(\hat{T}_3 - 2\hat{V}_3\right)$ satisfies $(\star_6 - i)G_3 = 0$, so that it is in the $(\star_6 + i)$-eigenspace. Therefore, this solution does not contain any piece of Hodge type $(3,0)$, even though $\hat{T}_3$ may have a non-vanishing singlet component $\hat{T}_{ijk}$ (see (3.1) for the action of $(\star_6 \pm i)$ on the various Hodge types of a 3-form). In this paper we will show, however, that when going beyond the linearised analysis of the equations of motion given in [4], a $(3,0)$-component of G_3 gets generated at order m^3. This will be summarised in the next section. This $(3,0)$-piece, which arises from the inhomogeneous solution of the equation of motion for G_3, can be derived from a 2-form potential with polarisation tensor ε_{ijk}, which is the polarisation tensor associated with the gaugino condensate $\langle\bar{\lambda}\bar{\lambda}\rangle$ in the confining phase, thereby linking the emergence of a $(3,0)$-piece in G_3 to the formation of a gaugino condensate.

3. The $(3,0)$-part of the 3-form G_3

At large r, the supergravity solution dual to the $\mathcal{N} = 1^*$ theory admits an integrable complex structure in the transverse 6-dimensional space. Locally, the associated fundamental

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2-form J may be written as $J = e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6$, and $(1,0)$-forms are given by $e^1 + ie^2$, $e^3 + ie^4$ and $e^5 + ie^6$. Given a 3-form G_3, we may then project onto the various Hodge types of G_3 by

$$\begin{aligned} (\star_6 - i) &\longrightarrow (3,0) + (1,2)_P + (2,1)_{NP}\,, \\ (\star_6 + i) &\longrightarrow (0,3) + (2,1)_P + (1,2)_{NP}\,, \end{aligned} \tag{3.1}$$

where the subscripts P and NP denote the primitive and non-primitive parts, respectively. Here, $\star_6$ is defined in terms of ϵ_{abcdef} with $a, \ldots = 1, \ldots, 6$.

To order m, the metric is given by

$$ds_{10}^2 = Z^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + Z^{1/2}g_{mn}dx^m dx^n\,, \tag{3.2}$$

where $g_{mn} = \delta_{mn}$. We may introduce complex coordinates as $e^1 + i\,e^2 = \sqrt{2}\,dz^1$, $e^3 + i\,e^4 = \sqrt{2}\,dz^2$ and $e^5 + i\,e^6 = \sqrt{2}\,dz^3$, where [4]

$$z^1 = \frac{x^4 + ix^7}{\sqrt{2}}\,, \qquad z^2 = \frac{x^5 + ix^8}{\sqrt{2}}\,, \qquad z^3 = \frac{x^6 + ix^9}{\sqrt{2}}\,. \tag{3.3}$$

To order m^3, the metric g_{mn} in the 6-dimensional transverse spaces ceases to be diagonal [29], and therefore the $(1,0)$-forms will define new complex coordinates v^1, v^2, v^3. The differentials dv^i, when expressed in terms of dz^i and $d\bar{z}^i$, may receive $d\bar{z}^i$-admixtures of order m^2. Therefore, what for instance is a $(2,1)$-form when expressed in terms of the differentials dz^i and $d\bar{z}^i$, may become a $(3,0)$-form when expressed in terms of the differentials dv^i. In order to be sure that we correctly identify the $(3,0)$-contribution, we have to first understand if there are spurious contributions to the $(3,0)$-part at order m^3 which come from the order m perturbation in G_3.

Since we will be working to order m^3, the $(3,0)$-part in question will be the one given in terms of differentials dv^i. We will, however, begin by using differentials dz^i and $d\bar{z}^i$. The 3-form flux $G_3 = G_{(1)} + G_{(3)}$ has order m-contributions $G_{(1)}$ as well as order m^3-contributions $G_{(3)}$. The order m-contribution is given by the $p = -4$ solution discussed in the previous section, $G_{(1)} = 3r^{-4}\,(T_3 - \frac{4}{3}V_3)$, and it contains a primitive $(2,1)$-part $G_P^{(2,1)}$ (in differentials dz^i and $d\bar{z}^i$). This is the only contribution in $G_{(1)}$ which may turn into a $(3,0)$-part (which we denote by $\Upsilon_v^{(3,0)}$) of order m^3 when expressed in terms of differentials dv^i. We therefore write

$$G_P^{(2,1)} = \Upsilon_v^{(3,0)} + \cdots\,. \tag{3.4}$$

As we will be showing in this paper, the contribution $G_{(3)}$ is given by the following expression,

$$G_{(3)} = Z\,U\,dz^1 \wedge dz^2 \wedge dz^3 - \frac{i}{2}(\tilde{\star}_6 - \star_6)G_P^{(2,1)} + \cdots\,, \tag{3.5}$$

where $\star_6$ denotes the dual with respect to the flat metric δ_{mn}, whereas $\tilde{\star}_6$ denotes the dual with respect to the curved (order m^2 corrected) metric g_{mn}. Using (3.4) and (3.1), we obtain $\star_6 G_P^{(2,1)} = iG_P^{(2,1)} = i(\Upsilon_v^{(3,0)} + \cdots)$ as well as $\tilde{\star}_6 G_P^{(2,1)} = \tilde{\star}_6(\Upsilon_v^{(3,0)} + \cdots) = -i\Upsilon_v^{(3,0)} + \cdots$, where the dots denote forms which are not of the $(3,0)$-type in dv^i-differentials. It follows that

$$G_{(3)} = Z\, U\, dv^1 \wedge dv^2 \wedge dv^3 - \Upsilon_v^{(3,0)} + \cdots \tag{3.6}$$

when expressed in terms of differentials dv^i. Adding up (3.4) and (3.6), we find that the $(3,0)$-part of G_3 (in differentials dv^i) is, to order m^3, given by

$$G^{(3,0)} = Z\, U\, dv^1 \wedge dv^2 \wedge dv^3 \,. \tag{3.7}$$

We find, moreover, that (3.7) is induced by the $(1,0)$-part of $d\tau$ of order m^2. This, as we will show, is in accordance with the supersymmetry analysis given in [34].

The $(3,0)$-piece (3.7) may be obtained from a 2-form potential with polarisation tensor ε_{ijk}, as follows. The function U is given by

$$U = \frac{1}{27g}\, m^3\, R^4\, r^{-6}\, \bar{z}^4 \,, \tag{3.8}$$

where $r^2 = 2\,\bar{z}^i z^i$ and here and in what follows we use the definitions $\bar{z}^2 \equiv \bar{z}^i \bar{z}^i$, $\bar{z}^4 = \left(\bar{z}^2\right)^2$. Note that to order m^3 we may simply use the coordinates z^i instead of the v^i. Then, it can be checked that $G^{(3,0)}$ may be written as

$$G^{(3,0)} = \partial Y^{(2,0)} = -\frac{1}{108}\, g^{-1}\, m^3\, R^8\, \partial\left(\frac{\bar{z}^4}{r^4}\, X_2\right) , \tag{3.9}$$

where X_2 denotes the following 2-form potential

$$X_2 = r^{-6}\, \varepsilon_{ijk}\, z^i dz^j \wedge dz^k \,, \tag{3.10}$$

which satisfies $\partial X_2 = 0$. Observe that the 2-form potential $Y^{(2,0)}$ is not a harmonic 2-form on S^5, but this was not to be expected anyway since $G^{(3,0)}$ does not arise as a solution to the linearised equations of motion.

The homogeneous $p = -6$ solution given in (2.9) may also contain a term with polarisation tensor ε_{ijk}, which when added to $Y^{(2,0)}$, results in a potential given by

$$C_2 - \tau_0 B_2 \,\sim\, m^3\, r^{-6}\left(c + \frac{\bar{z}^4}{r^4}\right) \varepsilon_{ijk}\, z^i\, dz^j \wedge dz^k \,, \tag{3.11}$$

where the constant c denotes the contribution from the $p = -6$ solution. The constant b appearing in (2.3) is then proportional to $m^3\, c$. Observe that the angular contribution $\bar{z}^4\, r^{-4}$ cannot be cancelled against the constant c of the homogeneous solution. Since in the confining phase the polarisation tensor ε_{ijk} is associated to the vev $\langle\bar{\lambda}\bar{\lambda}\rangle$, we conclude that the $(3,0)$-part of G_3 contributes to the vev of the gaugino condensate $\bar{\lambda}\bar{\lambda}$. The Higgs and confining phases are related by $g \leftrightarrow 1/g$ [4]. In both phases there is the 2-form potential with polarisation tensor ε_{ijk}. In the Higgs phase this is related to the vev of $\mathrm{tr}\,\bar{\Phi}^3$.

Actually, the above doesn't uniquely determine the potential associated to $G^{(3,0)}$, since $G^{(3,0)}$ may also be written as

$$G^{(3,0)} = -\frac{1}{432}\, g^{-1}\, m^3\, R^8\, \partial\left(\frac{\bar{z}^2}{r^8}\, \varepsilon_{\bar{\imath}jk}\, \bar{z}^i dz^j \wedge dz^k\right) , \tag{3.12}$$

which results in a 2-form potential $P^{(2,0)}$ with a polarisation tensor $\varepsilon_{\bar{i}\bar{j}k}$. The complex conjugate tensor $\varepsilon_{ij\bar{k}}$ is proportional to $T_{i\bar{j}\bar{k}}$ and therefore associated to the mass deformation $W = \frac{1}{2} m_{ij}\, \Phi_{(i}\Phi_{j)}$ (note that it cannot be associated to a vev of a bilinear in the matter fermions, since such a vev would break supersymmetry).

Thus, the 2-form potential associated with $G^{(3,0)}$ is a linear combination of the potentials $Y^{(2,0)}$ and $P^{(2,0)}$. In order to determine the precise linear combination, one would have to fully determine G_3 at order m^3, which we haven't done, and to derive it from a 2-form potential. So, even though we cannot at this stage determine the precise linear combination, the above shows that $G^{(3,0)}$ can be associated to the vev of the gaugino condensate.

4. Determining $G^{(3,0)}$

4.1 Bulk contribution to τ

It will turn out that the $(3,0)$-part of G_3 at order m^3 is driven by a non-constant axion/dilaton τ at order m^2. In this section we will therefore determine the dependence of τ on the six transverse coordinates x^m at order m^2. The non-trivial behaviour of the dilaton $\Phi(x^m)$ has already been determined in [29].

To linear order in m, both the dilaton Φ and the axion C are constant [4]. To quadratic order in m this will not any longer be the case.

The bulk equation of motion for the dilaton reads [4]

$$\nabla^M \nabla_M \Phi = \mathrm{e}^{2\Phi}\, \partial_m C \partial^m C + \frac{g\mathrm{e}^{\Phi}}{12} \mathrm{Re}\, (G_{mnp} G^{mnp})\,. \tag{4.1}$$

The bulk equation of motion for C [4],

$$\nabla^M (\mathrm{e}^{2\Phi} \partial_M C) = -\frac{g\mathrm{e}^{\Phi}}{6} H_{mnp} \tilde{F}^{mnp}\,, \tag{4.2}$$

shows that $\partial_m C$ is non-vanishing to second order in m, i.e. $\partial_m C \sim m^2$. Therefore $\partial_m C \partial^m C \sim m^4$, and we can neglect this term in (4.1) relative to $G_{mnp}G^{mnp}$, which is of order m^2.

Using the bulk expression for G_3 [4], $G_3 = d\eta_2 = g^{-1}\, Z\, (T_3 - \frac{4}{3} V_3)$, as well as (3.2) we compute [29]

$$G_{mnp}\, G^{mnp} = g^{-2} Z^{1/2} \left(T_{mnp}\, T_{mnp} - \frac{8}{3} \frac{x^m x^n}{r^2}\, T_{mqp}\, T_{nqp} \right), \tag{4.3}$$

where the contractions on the right hand side are with respect to the flat metric δ_{mn}. Using the definition of T_3

$$T_3 = m\, \left(dz^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3 + d\bar{z}^1 \wedge dz^2 \wedge d\bar{z}^3 + d\bar{z}^1 \wedge d\bar{z}^2 \wedge dz^3 \right), \tag{4.4}$$

which has Hodge type $(1,2)$ and is primitive, i.e. $T_3 \wedge J = 0$, we get

$$G_{mnp}\, G^{mnp} = \frac{32 i}{3}\, g^{-2}\, Z^{1/2}\, m^2\, \left(\Sigma + i \frac{Y}{2} \right). \tag{4.5}$$

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Here Σ and Y denote two SO(6) scalar harmonics satisfying

$$\Delta_{\rm flat} H = -\frac{12}{r^2} H\,, \qquad H = \Sigma, Y\,, \tag{4.6}$$

where $\Delta_{\rm flat}$ is the flat laplacian in six dimensions. Σ is given by

$$\Sigma\left(\frac{x^m}{r}\right) = \frac{x^4x^7 + x^5x^8 + x^6x^9}{r^2} = -i\,\frac{[z^2 - \bar{z}^2]}{2r^2}\,, \tag{4.7}$$

and Y by

$$Y\left(\frac{x^m}{r}\right) = \frac{\sum_{i=4}^{6}(x^i)^2 - \sum_{i=7}^{9}(x^i)^2}{r^2} = \frac{z^2 + \bar{z}^2}{r^2}\,, \tag{4.8}$$

where $r^2 = 2\,z^i\bar{z}^i$ and $z^2 = z^iz^i$.

The bulk equations of motion (4.1) and (4.2) can be solved at large r, where $Z = R^4/r^4$. To order m^2, the bulk equation of motion (4.1) becomes

$$Z^{-1/2}\,\Delta_{\rm flat}\Phi = \frac{g^2}{12}{\rm Re}\left(G_{mnp}G^{mnp}\right), \tag{4.9}$$

and it has been solved in [29]. This equation has two homogeneous solutions $\Phi = r^2Y$, $r^{-6}Y$, which are the non-normalisable and normalisable solutions for a field operator of scale dimension $\Delta = 6$ (see (2.3)). In the list of [40], these homogeneous solutions correspond to massive spin zero $k = 2$ deformations in the $\mathbf{20_c}$ representation of SO(6) ($M^2 = k(k+4) = \Delta(\Delta-4) = 12$ in the notation of [40]). These homogeneous solutions will not be turned on in the following.

Equation (4.9) also has an inhomogeneous solution, which at large r reads[4] [29]

$$\Phi - \Phi_0 = \frac{m^2R^4}{36r^2}\,Y\left(\frac{x^m}{r}\right) = \frac{m^2R^4}{36}\,\frac{\left[z^2 + \bar{z}^2\right]}{r^4}\,. \tag{4.10}$$

Note that Φ has mass dimension zero, as it should. It follows that

$$d\Phi|_{(1,0)} = \frac{m^2R^4}{18r^4}\left(z^j - 2\,\frac{\bar{z}^j}{r^2}\left[z^2 + \bar{z}^2\right]\right)dz^j\,. \tag{4.11}$$

The bulk equation of motion (4.2) can, to order m^2, be written as

$$Z^{-1/2}\,\Delta_{\rm flat}C = \frac{g}{6}\,{\rm Im}G_{mnp}\,{\rm Re}G^{mnp}\,. \tag{4.12}$$

Using (4.5) and inserting the ansatz $C = f(r)\,\Sigma$ into (4.12) yields

$$\left(\frac{d^2}{dr^2} + \frac{5}{r}\frac{d}{dr} - \frac{12}{r^2}\right) f(r) = \frac{8\,g^{-1}\,m^2\,R^4}{9\,r^4}\,. \tag{4.13}$$

This has two homogeneous solutions of the form $f = r^2, r^{-6}$, which are the non-normalisable and normalisable solutions for a field operator of scale dimension $\Delta = 6$. Again, these solutions will not be considered in the following.

[4]Note that (4.10) differs by a factor 18 from the result given in [29].

Equation (4.13) also has an inhomogeneous solution given by

$$f(r) = -\frac{g^{-1}\, m^2\, R^4}{18\, r^2}\,, \tag{4.14}$$

yielding

$$g\,(C - C_0) = -\frac{m^2\, R^4}{18\, r^2}\,\Sigma\,. \tag{4.15}$$

Then, using $\tau = C + i\mathrm{e}^{-\Phi}$ and $\tau_0 = C_0 + ig^{-1}$, we find

$$g\,(\tau - \tau_0) = -\frac{m^2\, R^4}{18\, r^2}\left(\Sigma + i\frac{Y}{2}\right) = -im^2\, R^4\,\frac{\bar z^2}{18\, r^4} \tag{4.16}$$

at large r.

The result (4.16) agrees with the analysis given in [31], where supersymmetry was used to determine its general form.

Thus, to order m^2, the inhomogeneous solution $\tau - \tau_0$ is a linear combination of the SO(6) scalar harmonics Σ and Y with eigenvalue $M^2 = 12$. This angular dependence is induced by $G^{mnp}G_{mnp}$.

In the following sections we will make use of the $(1,0)$-part of (4.16), which reads

$$g\, d\tau|_{(1,0)} = \frac{2i}{9}\, m^2 R^4 \frac{\bar z^2}{r^6}\, \bar z^j dz^j\,. \tag{4.17}$$

4.2 Source terms

We now discuss the contributions to G_3 due to the presence of sources and argue that we can neglect them in the computation of $G^{(3,0)}$.

As shown in [4], the Bianchi identity for G_3 gets modified by a magnetic source J_4, $dG_3 = J_4$, which is a source for a D5/NS5-brane solution. The latter is induced by the Myers effect. The source term J_4 is proportional to $M\,\delta(w - r_0)$. Here the coordinates w^i are the imaginary part of the z^i, i.e. x^7, x^8, x^9. The quantity M is given by $M = c\,\tau_0 + d$. In the Higgs phase $(c,d) = (0,1)$, so that $M = 1$, whereas in the confining phase $(c,d) = (1,0)$, so that $M = \tau_0$, where we may take $\tau_0 = ig^{-1}$. Therefore, in the Higgs phase, J_4 is real and hence $H_3 = 0$, whereas in the confining phase J_4 is imaginary and hence $F_3 = 0$. In the Higgs phase the D3-branes expand to a D5-brane with topology $R^4 \times S^2$, where the 2-sphere S^2 has radius $r_0 \sim m$. The confining vacuum, on the other hand, is described by a NS5-brane solution of the same topology and with radius $r_0 \sim g\, m$ [4]. Thus, J_4 has indeed the expected behaviour of a source for a D5/NS5-brane solution.

The source modification of the Bianchi identity for G_3 results in a modification of the sourceless order m solution for G_3, namely [4] $G_3 = d\eta_2 \to G_3 = d\eta_2 + (\star_6 + i)d\omega_2$. Both η_2 and ω_2 now depend on r_0. Even though ultimately the two parameters r_0 and m become linked, as discussed above, it is helpful to think of r_0 and m as independent parameters in which one may power expand the solution. Therefore, the solution constructed in [4] is valid to linear order in m and to any order in r_0. This modification of G_3 is in agreement [31] with supersymmetry, since it is of the form (A.5), with $Z = R^4/r^4$ replaced by $Z = R^4/(AB)$,

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where $A = y^2 + (w + r_0)^2$ and $B = y^2 + (w - r_0)^2$, and where the coordinates y^i denote x^4, x^5, x^6. Observe that at large r, $d\eta_2$ goes as r^{-4}, whereas $d\omega_2$ behaves as r^{-6}. Therefore, asymptotically $d\omega_2$ is subleading relative to $d\eta_2$.

Even though both η_2 and ω_2 now depend on r_0, $Z^{-1}(\star_6 - i)G_3$ is still constant and given by $Z^{-1}(\star_6 - i)G_3 = Z^{-1}(\star_6 - i)d\eta_2 = -\frac{2i}{3}g^{-1}T_3$ [4]. Therefore, $(\star_6 - i)G_3$ is still of type $(1, 2)$ and primitive. Thus, to linear order in m and to any order in r_0, there are no terms in G_3 of Hodge type $(3, 0)$.

A term in G_3 of type $(3, 0)$ may arise from bulk effects. In the following, we will show that bulk effects do give rise to a $(3, 0)$-term of order m^3. In doing so, we will be ignoring corrections due to sources, i.e. we will be neglecting r_0-corrections. This is consistent, since we will only be interested in the asymptotic (large r) behaviour of the induced $(3, 0)$-term in G_3.

Observe that when solving the equations of motion for τ to order m^2, we also neglected the presence [4] of source terms in the equation of motion for τ. This is again consistent in the large r limit, where the corrections due to the sources are subleading.

4.3 Bulk contribution to G_3

The 3-form G_3 has been determined in [4] to linear order in m and in the presence of a magnetic source J_4. Here, we will determine some of the contributions to G_3 arising at order m^3. As explained previously, we will neglect r_0-contributions from the magnetic source, which are subleading.

To order m^3, the bulk equation of motion for G_3 is not any longer given by (2.10), but there are additional terms present which are quadratic in the fluctuating fields, as follows. From [4, equation (18)] we can read off the bulk equation of motion for $G_3 = F_3 - \tau H_3$,

$$d \star G_3 = igF_5 \wedge G_3 - ie^{\Phi} d\tau \wedge \star \tilde{F}_3 \,. \tag{4.18}$$

Here and in what follows $\tilde{F}_3 = F_3 - CH_3$ and $\tilde{F}_5 = F_5 - C_2 \wedge H_3$. Using $F_5 \wedge G_3 = \tilde{F}_5 \wedge G_3$ (valid whenever G_3 has only non-vanishing legs in the 6-dimensional transverse space), we obtain

$$d \star G_3 = ig\tilde{F}_5 \wedge G_3 - ie^{\Phi} d\tau \wedge \star \mathrm{Re}\, G_3 \,. \tag{4.19}$$

The five-form $\tilde{F}_5$ is given by $\tilde{F}_5 = (1+\star)\, d\chi_4$, where $\chi_4 = g^{-1}Z^{-1}\, D(r)\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. Inspection of [29, equation (105)] shows that $D(r)$ receives a correction to order m^2. At large r, $D(r)$ is given by $D(r) = 1 + (7m^2R^4r^{-2})/3$. The explicit form of $D(r)$ will, however, not be needed in the following. Thus we obtain

$$g\tilde{F}_5 \wedge G_3 = d(Z^{-1}D(r)) \wedge G_3 \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \,. \tag{4.20}$$

The ten-dimensional line element corrected by m^2 terms reads

$$ds^2 = Z^{-\frac{1}{2}}\, A(r)\, \eta_{\mu\nu}\, dx^\mu dx^\nu + Z^{\frac{1}{2}}\, g_{mn}\, dx^m dx^n \,. \tag{4.21}$$

At order m^2, the metric g_{mn} ceases to be diagonal, and $A(r)$ receives a correction as well, which at large r reads $A(r) = 1 + (7m^2R^4r^{-2})/24$ [29]. The explicit form of $A(r)$ will, however, not be needed in the following.

For a metric of the form (4.21) we have

$$\star G_3 = Z^{-1}A^2(r)(\tilde{\star}_6 G_3) \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \,, \tag{4.22}$$

where $\tilde{\star}_6$ denotes the dual in the 6-dimensional transverse space with respect to the corrected metric g_{mn}. The dual with respect to the flat 6-dimensional metric δ_{mn} will be denoted by $\star_6$. We then obtain from (4.19) that

$$d\left(Z^{-1}A^2(r)\tilde{\star}_6 G_3\right) = id\left(Z^{-1}D(r)\right) \wedge G_3 - i\mathrm{e}^{\Phi} d\tau \wedge \tilde{\star}_6 \left(Z^{-1}A^2(r)\,\mathrm{Re}\,G_3\right). \tag{4.23}$$

In the presence of a non-trivial axion/dilaton field τ, the bulk Bianchi identity for G_3 reads

$$dG_3 = -d\tau \wedge H_3 = -\frac{i}{2}\, g\, d\tau \wedge (G_3 - \bar{G}_3)\,. \tag{4.24}$$

Thus we may rewrite (4.23) as

$$\begin{aligned} d\left[Z^{-1}A^2(r)(\tilde{\star}_6 - i)G_3\right] &= \mathrm{id}\left[Z^{-1}(D(r) - A^2(r))\right] \wedge G_3 + iZ^{-1}A^2(r)d\tau \wedge H_3 - \\ &\quad - i\mathrm{e}^{\Phi} d\tau \wedge \tilde{\star}_6 \left(Z^{-1}A^2(r)\,\mathrm{Re}\,G_3\right) \\ &= \mathrm{id}\left[Z^{-1}(D(r)-A^2(r))\right] \wedge G_3 - \mathrm{e}^{\Phi} Z^{-1}A^2(r)d\tau \wedge (\tilde{\star}_6 + i)\mathrm{Im}\,G_3 - \\ &\quad - i\mathrm{e}^{\Phi} Z^{-1}A^2(r)d\tau \wedge \tilde{\star}_6\, G_3\,. \end{aligned} \tag{4.25}$$

We note that both $D - A^2$ and $d\tau$ are of order m^2. We solve (4.25) to order m^3 by inserting the order m value for G_3 on the rhs of (4.25). To order m^3 (4.25) becomes

$$\begin{aligned} d\left[Z^{-1}A^2(r)(\tilde{\star}_6 - i)G_3\right] &= id\left[Z^{-1}(D(r) - A^2(r))\right] \wedge G_3 - \\ &\quad - g\, Z^{-1}\, d\tau \wedge \left[(\star_6 + i)\mathrm{Im}\,G_3 + i\star_6\, G_3\right]. \end{aligned} \tag{4.26}$$

The order m value for G_3 reads [4] $G_3 = (\star_6 + i)d\omega_2 + d\eta_2$. The ω_2-contribution stems from the inclusion of a magnetic source in the Bianchi identity for G_3. At large r, $d\omega_2$ is subleading relative to $d\eta_2$. We therefore only keep the $d\eta_2$-terms in G_3,

$$G_3 \approx d\eta_2 = g^{-1}\, Z\left(T_3 - \frac{4}{3}V_3\right). \tag{4.27}$$

In the appendix we have listed various properties of T_3 and V_3. Both T_3 and V_3 are written in terms of differentials dz^i and $d\bar{z}^i$. Observe that to order m, there are no $(3,0)$ and $(2,1)_{NP}$ parts in (4.27).

Using $dr \wedge V_3 = 0$ and (A.2), we obtain from (4.26)

$$\begin{aligned} d\left[Z^{-1}A^2(r)\left((\star_6 - i)\, G_3 + (\tilde{\star}_6 - \star_6)G_3\right)\right] &= i\, g^{-1} Z\, d\left[Z^{-1}(D(r) - A^2(r))\right] \wedge T_3 - \\ &\quad - \frac{1}{3}\, d\tau \wedge \left(T_3 - \bar{T}_3\right). \end{aligned} \tag{4.28}$$

Observe that $(\tilde{\star}_6 - \star_6)$ is of order m^2.

We will now determine the $(3,0)$-part of the flux G_3 to order m^3. To do so, we will be interested in the $(3,1)$-part of equation (4.28). On the right hand side, only the term proportional to $d\tau \wedge \bar{T}_3$ contributes. On the left hand side, the following terms may contribute. From (3.1) we establish $(\star_6 - i)\, G_3 = -2i\,(G^{(3,0)} + G^{(2,1)}_{NP} + G^{(1,2)}_{P})$. To order

m, the only non-vanishing piece in $(\star_6 - i)\, G_3$ is the $G_P^{(1,2)}$-part. Since $(A^2 - 1)\, G_P^{(1,2)}$ is of type $(1,2)$, it will not contribute to the $(3,1)$-part of equation (4.28). However, to order m^3, new pieces $G_{(3)}^{(3,0)}$ and $G_{(3)NP}^{(2,1)}$ may get generated, and they will contribute to the $(3,1)$-part of (4.28) (a new $G_{(3)P}^{(1,2)}$ may also be generated, but if we assume that the 6-dimensional transverse space is complex, then this will not contribute to the $(3,1)$-part of (4.28)). Now consider the contribution $(\tilde{\star}_6 - \star_6)\, G_3$. To order m, the $(2,1)_P$-part of G_3 is non-vanishing, whereas the $(2,1)_{NP}$-part vanishes. Therefore, the only term in $(\tilde{\star}_6 - \star_6)\, G_3$ which may contain a $(3,0)$-piece to order m^3 is the term $(\tilde{\star}_6 - \star_6) G_P^{(2,1)}$. To order m, there is also a non-vanishing $(1,2)$-part $G^{(1,2)}$. Therefore, $(\tilde{\star}_6 - \star_6)(G_P^{(2,1)} + G^{(1,2)})$ may contribute $(2,1)$-parts to order m^3. Hence we infer from (4.28) that to order m^3

$$-2i\, d\left[Z^{-1}\left(G_{(3)}^{(3,0)} + G_{(3)NP}^{(2,1)} + \frac{i}{2}\,(\tilde{\star}_6 - \star_6)\left(G_P^{(2,1)} + G^{(1,2)}\right)\right)\right]\Bigg|_{(3,1)} = \frac{1}{3}\, d\tau|_{(1,0)} \wedge \bar{T}_3\,. \tag{4.29}$$

Equation (4.29) may be solved as follows. Using $d = \partial + \bar{\partial}$, we write $\bar{T}_3 = \bar{\partial}\bar{S}_2^{(2,0)}$, where

$$\bar{S}_2^{(2,0)} = \frac{m}{2}\,\varepsilon_{\bar{i}jk}\,\bar{z}^i\, dz^j \wedge dz^k\,. \tag{4.30}$$

Then, assuming that the 6-dimensional transverse space is complex, we obtain

$$d\tau|_{(1,0)} \wedge \bar{T}_3 = -\bar{\partial}\left(\partial\tau \wedge \bar{S}_2^{(2,0)}\right) - \partial\left(\bar{\partial}\tau \wedge \bar{S}_2^{(2,0)}\right). \tag{4.31}$$

Introducing U as

$$G_{(3)}^{(3,0)} + \frac{i}{2}\left((\tilde{\star}_6 - \star_6)\, G_P^{(2,1)}\right)\Big|_{(3,0)} = Z\, U\, dz^1 \wedge dz^2 \wedge dz^3 \tag{4.32}$$

and $V = V_i\, dz^i$ as

$$G_{(3)NP}^{(2,1)} + \frac{i}{2}\left((\tilde{\star}_6 - \star_6)\left(G_P^{(2,1)} + G^{(1,2)}\right)\right)\Big|_{(2,1)_{NP}} = -i\, Z\, V \wedge J\,, \tag{4.33}$$

the l.h.s. of (4.29) can be written as

$$-2i\bar{\partial}\left(U dz^1 \wedge dz^2 \wedge dz^3\right) - 2\partial\,(V \wedge J) + \partial\left[Z^{-1}\left((\tilde{\star}_6 - \star_6)\left(G_P^{(2,1)} + G^{(1,2)}\right)\right)\Big|_{(2,1)_P}\right]. \tag{4.34}$$

Then, comparing (4.34) with (4.31) yields

$$U dz^1 \wedge dz^2 \wedge dz^3 = -\frac{i}{6}\,\partial\tau \wedge \bar{S}_2^{(2,0)}\,, \tag{4.35}$$

as well as

$$V \wedge J = \frac{1}{6}\left(\bar{\partial}\tau \wedge \bar{S}_2^{(2,0)}\right)\Big|_{NP}\,, \tag{4.36}$$

and also

$$Z^{-1}\left((\tilde{\star}_6 - \star_6)\left(G_P^{(2,1)} + G^{(1,2)}\right)\right)\Big|_{(2,1)_P} = -\frac{1}{3}\left(\bar{\partial}\tau \wedge \bar{S}_2^{(2,0)}\right)\Big|_P\,, \tag{4.37}$$

up to various integration functions which we have set to zero. The subscripts NP and P denote the non-primitive and primitive parts, respectively.

Observe that (4.32) is of the form (3.5). It then follows from (3.7) that ZU is the total induced $(3,0)$-form at order m^3. We therefore have

$$G^{(3,0)} = ZU\, dz^1 \wedge dz^2 \wedge dz^3 = -\frac{i}{6}\, Z\, \bar{S}_2^{(2,0)} \wedge d\tau|_{(1,0)}\,. \tag{4.38}$$

Inserting (4.17) into (4.38) yields

$$U = \frac{1}{27}\, g^{-1} m^3 R^4\, \frac{\bar{z}^2\, \bar{z}^2}{r^6}\,. \tag{4.39}$$

In the next section we will show that $G^{(3,0)}$ is in precise agreement with supersymmetry.

Using

$$\left(\bar{\partial}\tau \wedge \bar{S}_2^{(2,0)}\right)\Big|_{NP} = \frac{g^{-1} m^3\, R^4}{9}\, \frac{\bar{z}^2}{r^6}\, \varepsilon_{s\bar{t}k}\, z^s \bar{z}^t\, dz^k \wedge J\,, \tag{4.40}$$

we infer from (4.36) that

$$V = \frac{g^{-1} m^3\, R^4}{54}\, \frac{\bar{z}^2}{r^6}\, \varepsilon_{s\bar{t}k}\, z^s \bar{z}^t\, dz^k\,. \tag{4.41}$$

And finally, we note that (4.37) must be satisfied for consistency. Although we did not perform this check explicitly, the alternative derivation of (4.35) using supersymmetry enforces (4.37).

To summarise, we have found that to order m^3, there is a $(3,0)$-piece in the flux G_3,

$$G^{(3,0)} = Z\, U\, dv^1 \wedge dv^2 \wedge dv^3 = -\frac{g^{-1} m^3\, R^8}{27\, r^6} \left(\Sigma + i\frac{Y}{2}\right)^2 dv^1 \wedge dv^2 \wedge dv^3\,, \tag{4.42}$$

which is induced by the $(1,0)$-part of $d\tau$ of order m^2. This is a bulk effect. The fact that $G^{(3,0)}$ is generated by the $(1,0)$-part of $d\tau$ is in accordance with the supersymmetry analysis based on the type D ansatz [34], to which we now turn.

5. Supersymmetry and SU(2)-structure

We now check that the $(3,0)$-part of G_3 given in (4.38) is consistent with supersymmetry. The general analysis for the supersymmetry conditions in type IIB based on the type D ansatz (1.2) was performed in [34]. This type D ansatz yields various restrictions on the allowed 3-form flux. Here we will check those conditions which are related to the generation of $G^{(3,0)}$ to order m^3. A check of the consistency of the Polchinski-Strassler background with supersymmetry to order m has been given in [31].

We will be mainly concerned with the dilatino equations. In particular, one of the supersymmetry conditions derived from this in [34] states that the $(3,0)$-part of the flux G_3 is entirely generated by part of the $(1,0)$-form $\partial\tau$. The precise relation (for the signature used here) is as follows [34]

$$\kappa\, g_{3,0}^{SU} = Z^{1/2}\, \frac{\bar{c}}{a}\, p_1\,, \tag{5.1}$$

where $\kappa\, g_{3,0}^{SU} = ig\, \mathrm{e}^{i\theta} g_{3,0}$ and $G^{3,0} = g_{3,0}\, \Omega$, with $\Omega = 2\sqrt{2}\, dz^1 \wedge dz^2 \wedge dz^3$. The coefficients a and c are two of the coefficients appearing in the type D supersymmetry spinor ansatz (1.2). In this ansatz, the two spinors η_- and χ_+ are related by[5] $\chi_+ = \frac{1}{2} Z^{1/4}\, w_i\, \gamma^i\, \eta_-$.

Here $w = w_m dx^m = w_i\, dz^i$ denotes a globally defined 1-form necessary for the existence of an SU(2)-structure. The coefficient p_1 is the one appearing in the decomposition

$$P = P_m dx^m = p_1 w_m dx^m + p_2 \bar{w}_m dx^m + \Pi_m dx^m\,, \tag{5.2}$$

where $P_m = f^2\, \partial_m B = -\mathrm{e}^{2i\theta}\, \partial_m \tau/(\tau - \bar{\tau})$ [32]. The one-form Π obeys $w \lrcorner \Pi = \bar{w} \lrcorner \Pi = 0$. Here we use the conventions of [31], where $ds^2 = Z^{-1/2} \eta_{\mu\nu} dx^\mu dx^\nu + g_{mn} dx^m dx^n$ and

$$g_{i\bar{j}} = g_{\bar{j}i} = Z^{1/2}\, \delta_{i\bar{j}}\,, \qquad \{\gamma^i, \gamma^{\bar{j}}\} = 2 g^{i\bar{j}}\,, \qquad G^j{}_{jk} = g^{j\bar{i}}\, G_{\bar{i}jk}\,. \tag{5.3}$$

In these conventions, w_i satisfies $w_i\, \bar{w}_{\bar{i}}\, \delta^{i\bar{i}} = 2$.

To linear order in m, the coefficients a, b and c appearing in (1.2) can be determined from the result for the supersymmetry spinor given in [31],

$$\epsilon(x^\mu, x^m) = Z^{-1/8}\, \varepsilon(x^\mu) \otimes \eta_- + \epsilon_1(x^\mu, x^m)\,, \tag{5.4}$$

where

$$\epsilon_1(x^\mu, x^m) = Z^{-1/8}\, \frac{\kappa}{24 S^2}\, (\partial_m \log Z)\, \varepsilon^*(x^\mu) \otimes \gamma^m\, G_{SU}\, \eta_+\,. \tag{5.5}$$

Here $Z = R^4/r^4$ and $S^2 = 16\, Z^{-1/2}\, r^{-2}$. As before, $\kappa\, G_{SU} = ig\, \mathrm{e}^{i\theta}\, G$, where $G = G_{mnp}\, \gamma^{mnp}$.

From (5.4) we infer that $a = Z^{-1/8}$. In the following, we will show that $b = 0$ and compute c, which we will turn out to be non-vanishing. This fact implies that the transverse 6-dimensional space possesses an SU(2)-structure [34]. We also note that since c becomes non-trivial at order m, in the $m \to 0$ limit we recover the type B spinor ansatz as expected. Moreover, the fact that $b = 0$ is compatible with the expectation that the dielectric configuration which underlies the supergravity solution requires a supersymmetry spinor satisfying the projector condition (1.3).

Using

$$\begin{aligned} \gamma_{ijk}\, \eta_+ &= Z^{3/4}\, \Omega_{ijk}\, \eta_- = Z^{3/4}\, 2\sqrt{2}\, \varepsilon_{ijk}\, \eta_-\,, \\ \gamma^{\bar{i}\bar{k}}\, \eta_+ &= \frac{1}{2}\, Z^{-1/4}\, \delta^{\bar{i}i}\, \delta^{\bar{k}k}\, \Omega_{ikl}\, \gamma^l\, \eta_- \end{aligned} \tag{5.6}$$

as well as

$$G\, \eta_+ = G_{\bar{i}\bar{j}\bar{k}}\, \gamma^{\bar{i}\bar{j}\bar{k}}\, \eta_+ + 6 G^{\bar{j}}{}_{\bar{j}\bar{k}}\, \gamma^{\bar{k}}\, \eta_+ \tag{5.7}$$

we obtain

$$\begin{aligned} \partial_m Z\, \gamma^m\, G\, \eta_+ &= \partial_l Z\, Z^{-3/4}\, G_{\bar{i}\bar{j}\bar{k}}\, \Omega_{ijk}\, \gamma^l \eta_- + 12 \partial_i Z\, G^{\bar{j}}{}_{\bar{j}\bar{k}}\, g^{i\bar{k}}\, \eta_+ + 6 \partial_{\bar{i}} Z\, G^{\bar{j}}{}_{\bar{j}\bar{k}}\, \gamma^{\bar{i}\bar{k}}\, \eta_+ \\ &= 2\sqrt{2}\, \partial_l Z\, Z^{-3/4}\, G_{\bar{i}\bar{j}\bar{k}}\, \varepsilon_{ijk}\, \gamma^l \eta_- + 12 \partial_i Z\, G^{\bar{j}}{}_{\bar{j}\bar{k}}\, g^{i\bar{k}}\, \eta_+ + \\ &\quad + 6\sqrt{2}\, \partial_{\bar{i}} Z\, Z^{-1/4}\, G^{\bar{j}}{}_{\bar{j}\bar{k}}\, \delta^{\bar{i}i}\, \delta^{\bar{k}k}\, \varepsilon_{ikl}\, \gamma^l\, \eta_-\,. \end{aligned} \tag{5.8}$$

[5] The factor of $Z^{1/4}$ is due to the fact that here we use γ-matrices satisfying (5.3).

Using (A.3) we find that $\partial_i Z\, G^{\bar{j}}{}_{\bar{j}\bar{k}}\, g^{i\bar{k}}\, \eta_+ \propto \bar{z}^i\, \varepsilon_{stk}\, z^s \bar{z}^t\, \delta^{i\bar{k}} = 0$. It follows that

$$\begin{aligned}\partial_m Z\, \gamma^m\, G\, \eta_+ &= -16\sqrt{2}\, g^{-1} Z^{1/4} m \frac{z^2}{r^2}\, \partial_l Z\, \gamma^l \eta_- - \\ &\quad - 16\sqrt{2}\, g^{-1} Z^{1/4} m\, \frac{(z^l \bar{z}^i - z^i \bar{z}^l)}{r^2}\, \partial_{\bar{i}} Z\, \gamma^l\, \eta_- \,. \end{aligned} \tag{5.9}$$

With $Z = R^4/r^4$ it follows that $\partial_{\bar{l}} Z = -4Zr^{-2}\bar{z}^l$, and hence we obtain

$$\partial_m Z\, \gamma^m\, G\, \eta_+ = 32\sqrt{2}\, g^{-1} Z^{5/4}\, m\, \frac{z^l}{r^2}\, \gamma^l\, \eta_- \,. \tag{5.10}$$

This is of the form

$$\partial_m Z\, \gamma^m\, G\, \eta_+ = 32\sqrt{2}\, g^{-1} Z^{5/4}\, R^{-1}\, m\, \chi_+ \,, \tag{5.11}$$

where $\chi_+ = \frac{1}{2} Z^{1/4}\, w_l\, \gamma^l\, \eta_-$ and $w_l = 2r^{-1} z^l$. The latter satisfies $w_l\, \bar{w}_{\bar{l}}\, \delta^{l\bar{l}} = 2$ [34].

It follows that

$$\epsilon_1 = \frac{\sqrt{2}}{12}\, \mathrm{i}\mathrm{e}^{i\theta}\, R\, m\, Z^{1/8}\, \varepsilon^*(x^\mu) \otimes \chi_+ \,, \tag{5.12}$$

and hence

$$b = 0\,, \qquad c = \frac{\sqrt{2}}{12}\, \mathrm{i}\mathrm{e}^{i\theta}\, R\, m\, Z^{1/8} \,. \tag{5.13}$$

Having identified the one-form w,

$$w = w_m\, dx^m = w_i\, dz^i = 2\frac{z^i}{r}\, dz^i \,, \tag{5.14}$$

we proceed to decompose P as in (5.2). Using (4.16) we compute

$$P = \frac{1}{36} m^2 R^4\, \mathrm{e}^{2i\theta} \left(-4\frac{\bar{z}^2}{r^6}\, \bar{z}^i dz^i + \left[2\frac{\bar{z}^i}{r^4} - 4\frac{\bar{z}^2}{r^6}\, z^i \right] d\bar{z}^i \right) . \tag{5.15}$$

Equating (5.15) with (5.2) gives

$$\Pi = \frac{1}{36} m^2 R^4\, \mathrm{e}^{2i\theta} \left(\left[-4\frac{\bar{z}^2}{r^6}\, \bar{z}^i - 2\tilde{p}_1 \frac{z^i}{r} \right] dz^i + \left[2\frac{\bar{z}^i}{r^4} - 2\tilde{p}_2 \frac{\bar{z}^i}{r} - 4\frac{\bar{z}^2}{r^6}\, z^i \right] d\bar{z}^i \right) , \tag{5.16}$$

where $p_{1,2} = \frac{1}{36} m^2 R^4\, \mathrm{e}^{2i\theta}\, \tilde{p}_{1,2}$. Demanding that $w \lrcorner \Pi = 0$ as well as $\bar{w} \lrcorner \Pi = 0$ yields

$$\begin{aligned}\tilde{p}_1 &= -\frac{4}{r^3}\, \frac{\bar{z}^2}{r^2}\, \frac{\bar{z}^2}{r^2} = \frac{4}{r^3} \left(\Sigma + i\frac{Y}{2} \right)^2 \,, \\ \tilde{p}_2 &= \frac{1}{r^3} \left(1 - 4\, \frac{\bar{z}^2}{r^2}\, \frac{z^2}{r^2} \right) , \end{aligned} \tag{5.17}$$

where we used $dz^i \lrcorner dz^j = 0, d\bar{z}^i \lrcorner d\bar{z}^j = 0$ as well as $dz^i \lrcorner d\bar{z}^j = \delta^{i\bar{j}}$. Inserting (5.17) back into (5.16) yields

$$\Pi = -\frac{1}{9} m^2 R^4\, \mathrm{e}^{2i\theta}\, \frac{\bar{z}^2}{r^6} \left(\left(\bar{z}^i - 2\frac{\bar{z}^2}{r^2}\, z^i \right) dz^i + \left(z^i - 2\frac{z^2}{r^2}\, \bar{z}^i \right) d\bar{z}^i \right) . \tag{5.18}$$

Now we are in position to check (5.1). Using (4.42), the lhs of (5.1) gives

$$2\sqrt{2}\,\kappa\, g_{3,0}^{SU} = -\frac{i}{27}\,\mathrm{e}^{i\theta}\, m^3\, Z^2\, r^2 \left(\Sigma + i\frac{Y}{2}\right)^2 . \tag{5.19}$$

The rhs of (5.1) yields

$$2\sqrt{2}\, Z^{1/2}\, \frac{\bar{c}}{a}\, p_1 = -\frac{i}{27}\,\mathrm{e}^{i\theta}\, m^3\, Z^2\, r^2 \left(\Sigma + i\frac{Y}{2}\right)^2 , \tag{5.20}$$

Comparing (5.19) with (5.20) we find a perfect agreement.

The $(3,0)$-part of G_3 is thus due to the term $p_1 w$ in (5.2). It is instructive to make the this manifest in (4.38). Using

$$g\, d\tau|_{(1,0)} = -2i\mathrm{e}^{-2i\theta}\left(p_1\, w + \Pi_i\, dz^i\right) \tag{5.21}$$

and $\bar{S}_2^{(2,0)} \wedge \Pi_i\, dz^i = 0$, we obtain

$$G^{(3,0)} = -\frac{1}{3}\,\mathrm{e}^{-2i\theta}\, g^{-1}\, Z\, p_1\, \bar{S}_2^{(2,0)} \wedge w = -\frac{1}{3}\,\mathrm{e}^{-2i\theta}\, g^{-1}\, Z\, m\, p_1\, r\, dz^1 \wedge dz^2 \wedge dz^3 . \tag{5.22}$$

Another of the supersymmetry conditions derived from the dilatino variation states that [34]

$$p_2 = \kappa\, Z^{-1/2}\, \frac{c}{\bar{a}}\, g_{21}^{\mathrm{SU}} , \tag{5.23}$$

where

$$g_{21}^{\mathrm{SU}} = \frac{1}{16} Z^{3/2}\, X_{mnp}\, G_{\mathrm{SU}}^{mnp} , \qquad X_{mnp} = \overline{K}_{[mn}\, w_{p]} , \tag{5.24}$$

where as before $\kappa\, G_{\mathrm{SU}} = ig\,\mathrm{e}^{i\theta}\, G$. Here K is a 2-form satisfying $K \wedge w = \Omega = \frac{1}{3}\sqrt{2}\varepsilon_{ijk} dz^i \wedge dz^j \wedge dz^k$ as well as $K_{ij}\bar{K}_{\bar{i}\bar{j}} = 8$ and also $w \lrcorner K = \bar{w} \lrcorner K = 0$ [34]. Using (5.14) we therefore establish that

$$K = \sqrt{2}\, \varepsilon_{ij\bar{k}} \frac{\bar{z}^k}{r} dz^i \wedge dz^j . \tag{5.25}$$

It follows that

$$\overline{K} \wedge w = 2\sqrt{2}\, \varepsilon_{\bar{i}\bar{j}k}\, \frac{z^k z^l}{r^2} d\bar{z}^i \wedge d\bar{z}^j \wedge dz^l , \tag{5.26}$$

which is indeed primitive [34], i.e. $J \wedge \overline{K} \wedge w = 0$. The non-vanishing components of X_{mnp} are therefore $X_{\bar{i}\bar{j}l}$. Hence $X_{mnp}\, G^{mnp} = X_{\bar{i}\bar{j}l}\, G^{\bar{i}\bar{j}l} = Z^{-3/2}\, X_{\bar{i}\bar{j}l}\, G_{ij\bar{l}}$. Using the expressions for $G_{ij\bar{l}}$ given in (A.3) yields

$$X_{\bar{i}\bar{j}l}\, G_{ij\bar{l}} = \frac{8\sqrt{2}}{3}\, g^{-1}\, m\, Z\, r^3\, \tilde{p}_2 . \tag{5.27}$$

Now we are in position to check (5.23). The rhs of (5.23) yields

$$\frac{\kappa}{16}\, \frac{c}{\bar{a}}\, Z\, X_{mnp}\, G_{SU}^{mnp} = \frac{1}{36}\, \mathrm{e}^{2i\theta}\, m^2\, R^4\, \tilde{p}_2 = p_2 , \tag{5.28}$$

so that the supersymmetry condition is indeed satisfied. And finally, we observe that equations (3.6) and (3.7) given in [34] and stemming from the gaugino variation are also satisfied.

Hence we conclude that to order m^3 the bulk effects we computed are consistent with supersymmetry based on the type D ansatz (1.2).

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6. Conclusions

In this paper we computed order m^3 modifications of the Polchinski-Strassler solution due to bulk effects. We showed, in particular, that a $(3,0)$-piece in G_3 gets generated at order m^3. We argued that the associated 2-form potential may contain a term with the polarisation tensor ε_{ijk} which, in the confining phase of the dual $\mathcal{N}=1^*$ gauge theory, is associated to the formation of a gaugino condensate, thereby linking the emergence of a $G^{(3,0)}$-piece to the formation of a gaugino condensate. We also showed that the this $G^{(3,0)}$-piece, computed from the bulk equation of motion for G_3, is consistent with the type D spinor ansatz introduced in [34]. The latter is based on the existence of a globally defined complex vector w, whose asymptotic form we determined. The existence of w implies that the transverse 6-dimensional space possesses an SU(2)-structure [34]. Thus, the Polchinski-Strassler solution is a concrete example of a background with such a structure.

The results provided here are a further step towards the complete supergravity solution dual to the $\mathcal{N}=1^*$ gauge theory. Understanding that the full solution possesses an SU(2) structure should be helpful in simplifying the metric and flux ansaetze needed in order to obtain the full solution. It should also be noted that a candidate for the complete metric is given in [42], where the authors uplifted the 5-dimensional flow solution of GPPZ [43]. Although they give the metric, they did not determine G_3. We hope that our results will be useful in achieving this. Since [42] is the uplift of the 5-dimensional GPPZ solution, it can describe the same physics as [4] if all the massive KK modes vanish along the flow. As discussed in section 4, some of these modes may however be turned on, since we have encountered them as homogeneous solutions of the equations of motion for G_3, with overall coefficients which are only fixed by the infrared physics.

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A. T_3 and V_3

Here we review some of the properties of $G_3 = d\eta_2 = \zeta\, g^{-1}\, Z\,(T_3 - \frac{4}{3}V_3)$ [4]. We will set $\zeta = 1$ in the following. It may be restored by rescaling g^{-1}.

T_3 is given by (4.4). It has Hodge type $(1,2)$ and is primitive, i.e. $T_3 \wedge J = 0$. It satisfies $3T_3 = dS_2$ with

$$\begin{aligned} S_2 = m\Big(& z^1 d\bar z^2 \wedge d\bar z^3 + \bar z^1 dz^2 \wedge d\bar z^3 + \bar z^1 d\bar z^2 \wedge dz^3 - \\ & - z^2 d\bar z^1 \wedge d\bar z^3 - \bar z^2 dz^1 \wedge d\bar z^3 - \bar z^2 d\bar z^1 \wedge dz^3 + \\ & + z^3 d\bar z^1 \wedge d\bar z^2 + \bar z^3 dz^1 \wedge d\bar z^2 + \bar z^3 d\bar z^1 \wedge dz^2 \Big). \end{aligned} \tag{A.1}$$

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V_3 is given by $V_3 = d\log r \wedge S_2 = (2r^2)^{-1} dr^2 \wedge S_2$. G_3 can then be written as $G_3 = (3g)^{-1} R^4\, d\left(r^{-4} S_2\right)$.

T_3 and V_3 satisfy the following relations,

$$
\begin{aligned}
\star_6\, T_3 &= -iT_3\,,\\
\star_6\, V_3 &= -i(T_3 - V_3)\,,\\
(\star_6 - i)\, V_3 &= -iT_3\,,\\
(\star_6 + i)\, V_3 &= -i(T_3 - 2V_3)\,,\\
(\star_6 - i)\, d\eta_2 &= -\frac{2i}{3}\, g^{-1}\, Z\, T_3\,,\\
(\star_6 + i)\, d\eta_2 &= \frac{4i}{3}\, g^{-1}\, Z\, \left(T_3 - 2V_3\right)\,,\\
(\star_6 + i) d\bar{\eta}_2 &= \frac{2i}{3}\, g^{-1}\, Z\, \bar{T}_3\,,\\
(\star_6 + i)\, \mathrm{Im}\, d\eta_2 &= \frac{2}{3}\, g^{-1}\, Z\, \left(T_3 - 2V_3\right) - \frac{1}{3}\, g^{-1}\, Z\, \bar{T}_3\,,\\
\star_6\, d\eta_2 &= \frac{i}{3}\, g^{-1}\, Z\, \left(T_3 - 4V_3\right)\,,\\
(\star_6 + i)\, \mathrm{Im}\, d\eta_2 + i \star_6 d\eta_2 &= \frac{1}{3}\, g^{-1}\, Z\, (T_3 - \bar{T}_3)\,.
\end{aligned}
\tag{A.2}
$$

Inspection of (3.1) and of (A.2) shows that V_3 has Hodge types $(1,2), (2,1)$ and $(0,3)$, i.e. $V_3 = V^{(1,2)} + V^{(2,1)} + V^{(0,3)}$. Using that $r^2 = 2z^i\bar{z}^i$ and $J = i(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 + dz^3 \wedge d\bar{z}^3)$, we compute $V_3 = d\log r \wedge S_2 = (2r^2)^{-1} dr^2 \wedge S_2$ and obtain

$$
\begin{aligned}
V^{(1,2)} &= \frac{1}{2}\, T_3 + \Delta^{(1,2)}_{NP}\,,\\
\Delta^{(1,2)}_{NP} &= -i\, \frac{m}{r^2}\, \varepsilon_{i\bar{j}\bar{k}}\, z^i\, \bar{z}^j\, d\bar{z}^k \wedge J\,,\\
V^{(2,1)}_P &= \frac{m}{r^2}\Big([(\bar{z}^1)^2 + (\bar{z}^2)^2]\, dz^1 \wedge dz^2 \wedge d\bar{z}^3 + \\
&\qquad + [(\bar{z}^1)^2 + (\bar{z}^3)^2]\, dz^1 \wedge d\bar{z}^2 \wedge dz^3 + \\
&\qquad + [(\bar{z}^2)^2 + (\bar{z}^3)^2]\, d\bar{z}^1 \wedge dz^2 \wedge dz^3 - \\
&\qquad - \bar{z}^1 \bar{z}^3\, dz^2 \wedge (dz^3 \wedge d\bar{z}^3 - dz^1 \wedge d\bar{z}^1) + \\
&\qquad + \bar{z}^1 \bar{z}^2\, dz^3 \wedge (dz^2 \wedge d\bar{z}^2 - dz^1 \wedge d\bar{z}^1) + \\
&\qquad + \bar{z}^2 \bar{z}^3\, dz^1 \wedge (dz^3 \wedge d\bar{z}^3 - dz^2 \wedge d\bar{z}^2) \Big)\,,\\
V^{(0,3)} &= m\, \frac{\sum_i (z^i)^2}{r^2}\, d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3\,.
\end{aligned}
\tag{A.3}
$$

We also note the following useful relations,

$$
\begin{aligned}
(\star_6 - i)\, \Delta^{(1,2)}_{NP} &= 0\,,\\
(\star_6 - i)\, V^{(1,2)} &= -iT_3\,.
\end{aligned}
\tag{A.4}
$$

The expressions (A.3) are in agreement with the findings of [31] based on supersymmetry. Namely, setting $W = m\,\sum_i (z^i)^2$, we have

$$
\begin{aligned}
&Z\,\Delta_{NP}^{(1,2)} \propto \varepsilon_{ijk}\,A_{ij}\,d\bar{z}^k\,, \qquad A_{ij} = \partial_{[i}W\,\partial_{j]}Z\,,\\
&Z\,V^{(0,3)} \propto \partial_i W\,\partial_{\bar{i}} Z\,d\bar{z}^1 \wedge d\bar{z}^2 \wedge d\bar{z}^3\,,\\
&Z\,V_P^{(2,1)} \propto S_{\bar{i}\bar{j}}\,, \qquad S_{\bar{1}\bar{1}} = \frac{mZ}{r^2}[(\bar{z}^2)^2 + (\bar{z}^3)^2]\,, \qquad S_{\bar{1}\bar{2}} = \frac{mZ}{r^2}\bar{z}^1\,\bar{z}^2\,.
\end{aligned}
\tag{A.5}
$$

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